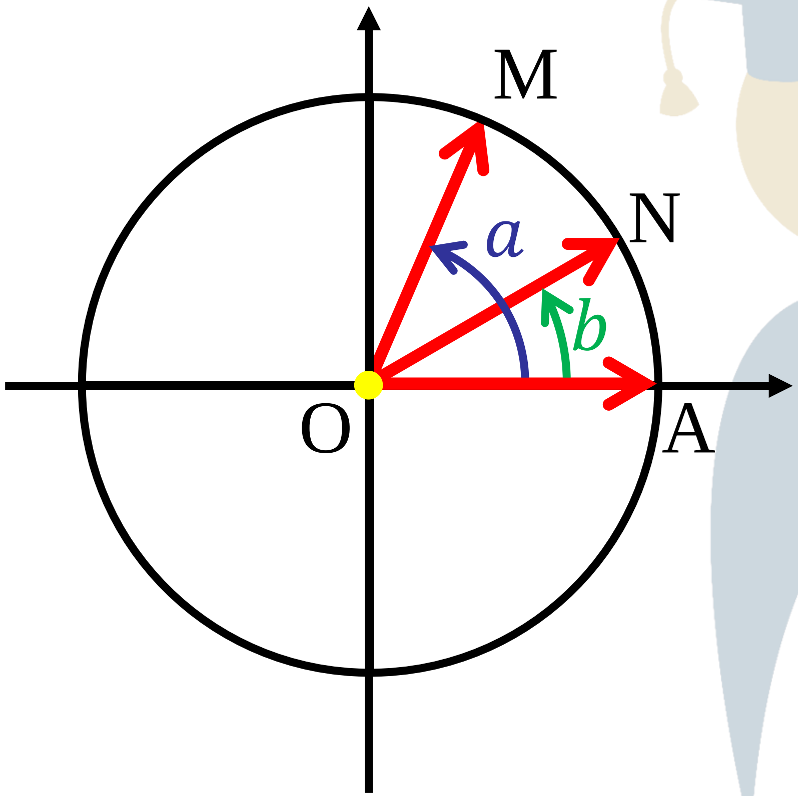
Trigonometric formulas

Addition formulas



$\overset{\frown}{AM}$ is an arc of measure a.

$\overset{\frown}{AN}$ is an arc of measure b.

The coordinates of M are $(\cos a\,;\sin a)$

The coordinates of N are $(\cos b\,;\sin b)$

$$\cos(a-b)=??$$

$$\cos(a-b)=\frac{\overrightarrow{OM}.\overrightarrow{ON}}{OM\times ON}=\frac{xx'+yy'}{1\times 1}$$

$$=\cos a\cos b+\sin a\sin b$$

$$\cos(a-b)=\cos a\cos b+\sin a\sin b$$

Addition formulas

$\cos(a+b) = ??$

$$\cos(a+b) = \cos\big(a-(-b)\big)$$
$$= \cos a\cos(-b) + \sin a \sin(-b)$$
$$= \cos a \cos b + \sin a\,(-\sin b)$$
$$= \cos a \cos b - \sin a \sin b$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

Addition formulas

$\sin(a+b) = ??$

$$\sin(a+b) = \cos\left(\frac{\pi}{2} - (a+b)\right)$$
$$= \cos\left(\left(\frac{\pi}{2} - a\right) - b\right)$$
$$= \cos\left(\frac{\pi}{2} - a\right)\cos b + \sin\left(\frac{\pi}{2} - a\right)\sin b$$
$$= \sin a \cos b + \cos a \sin b$$

$\sin(a-b) = ??$

$$\sin(a-b) = \sin\big(a + (-b)\big)$$
$$= \sin a \cos(-b) + \cos a \sin(-b)$$
$$= \sin a \cos b + \cos a\,(-\sin b)$$
$$= \sin a \cos b - \cos a \sin b$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$

Addition formulas

$\tan(a+b) = ??$

$$\tan(a+b) = \frac{\sin(a+b)}{\cos(a+b)} = \frac{\sin a \cos b + \cos a \sin b}{\cos a \cos b - \sin a \sin b}$$

$$= \frac{\cos a \cos b(\tan a + \tan b)}{\cos a \cos b(1 - \tan a \tan b)} = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$\tan(a-b) = ??$

$$\tan(a-b) = \tan\big(a + (-b)\big) = \frac{\tan a + \tan(-b)}{1 - \tan a \tan(-b)}$$

$$= \frac{\tan a - \tan b}{1 + \tan \tan b}$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

Example

$$\cos\left(\frac{\pi}{12}\right) = ??$$

$$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

$$\tan\left(\frac{\pi}{12}\right) = ??$$

$$\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{4}} = \frac{\sqrt{3}-1}{1+\sqrt{3}\times 1} =$$

$$\frac{\sqrt{3}-1}{1+\sqrt{3}} = \frac{4-2\sqrt{3}}{-2} = -2 + \sqrt{3}$$

$$\sin\left(\frac{\pi}{12}\right) = ??$$

$$\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\cos\left(\frac{11\pi}{12}\right) = ??$$

$$\cos\left(\frac{11\pi}{12}\right) = \cos\left(\pi - \frac{\pi}{12}\right) =$$

$$-\cos\frac{\pi}{12} = -\frac{\sqrt{2}+\sqrt{6}}{4}$$

Double angle formulas

$\cos(2a) = ??$

$$\cos(2a) = \cos(a + a)$$
$$= \cos a \cos a - \sin a \sin a$$
$$= \cos^2 a - \sin^2 a$$
$$= \cos^2 a - (1 - \cos^2 a)$$
$$= \cos^2 a - 1 + \cos^2 a$$
$$= 2\cos^2 a - 1 \quad \Rightarrow \quad \cos^2 a = \frac{1+\cos 2a}{2}$$
$$= 2(1 - \sin^2 a) - 1$$
$$= 2 - 2\sin^2 a - 1$$
$$= 1 - 2\sin^2 a \quad \Rightarrow \quad \sin^2 a = \frac{1-\cos 2a}{2}$$

Double angle formulas

$\sin(2a) = ??$

$\sin(2a) = \sin(a + a)$
$= \sin a \cos a + \cos a \sin a$
$= 2 \sin a \cos a$

$\tan(2a) = ??$

$\tan(2a) = \tan(a + a)$
$= \frac{\tan a + \tan a}{1 - \tan a \tan a} = \frac{2 \tan a}{1 - \tan^2 a}$

Example

Consider a real number a such that $\cos a = -\frac{3}{5}$

Calculate $\cos(2a)$; $\sin(2a)$ and $\tan(2a)$

$$\cos(2a) = 2\cos^2 a - 1 = 2\left(\frac{9}{25}\right) - 1 = -\frac{7}{25}$$

$$\sin^2(2a) = 1 - \cos^2(2a) = 1 - \frac{49}{625} = \frac{576}{625} \text{ so}$$

$$\sin(2a) = \pm\sqrt{\frac{576}{625}} = \pm\frac{24}{25}$$

$$\tan(2a) = \frac{\sin(2a)}{\cos(2a)} = \frac{\pm\frac{24}{25}}{-\frac{7}{25}} = \pm\frac{24}{7}$$

Factoring or Linearization

$\cos p + \cos q = ??$

Let $p = a + b$ and $q = a - b$

So $a = \frac{p+q}{2}$ & $b = \frac{p-q}{2}$

$\cos p + \cos q = \cos(a+b) + \cos(a-b)$

$= \cos a \cos b - \sin a \sin b + \cos a \cos b + \sin a \sin b$

$= 2\cos a \cos b = 2\cos\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right)$

$\cos p - \cos q = ??$

$\cos p - \cos q = \cos(a+b) - \cos(a-b)$

$= \cos a \cos b - \sin a \sin b - (\cos a \cos b + \sin a \sin b)$

$= \cos a \cos b - \sin a \sin b - \cos a \cos b - \sin a \sin b$

$= -2\sin a \sin b = -2\sin\left(\frac{p+q}{2}\right)\sin\left(\frac{p-q}{2}\right)$

Factoring or Linearization

$\sin p + \sin q = ??$

Let $p = a + b$ and $q = a - b$

So $a = \frac{p+q}{2}$ & $b = \frac{p-q}{2}$

$\sin p + \sin q = \sin(a+b) + \sin(a-b)$

$= \sin a \cos b + \cos a \sin b + \sin a \cos b - \cos a \sin b$

$= 2 \sin a \cos b = 2 \sin(\frac{p+q}{2}) \cos(\frac{p-q}{2})$

$\sin p - \sin q = ??$

$\sin p - \sin q = \sin(a+b) - \sin(a-b)$

$= \sin a \cos b + \cos a \sin b - (\sin a \cos b - \cos a \sin b)$

$= \sin a \cos b + \cos a \sin b - \sin a \cos b + \cos a \sin b$

$= 2 \cos a \sin b = 2 \cos(\frac{p+q}{2}) \sin(\frac{p-q}{2})$

Example

Solve the equation: $\sin x + \sin 2x + \sin 3x = 1 + \cos x + \cos 2x$

$\sin x + \sin 2x + \sin 3x = \sin(2x) + 2\sin 2x \cos x = \sin 2x\,(1 + 2\cos x)$

$1 + \cos x + \cos 2x = \cos x + 2\cos^2 x = \cos x\,(1 + 2\cos x)$

$\sin 2x\,(1 + 2\cos x) = \cos x\,(1 + 2\cos x)$

$\sin 2x\,(1 + 2\cos x) - \cos x\,(1 + 2\cos x) = 0$

$(1 + 2\cos x)(\sin 2x - \cos x) = 0$

$1 + 2\cos x = 0$ or $\sin 2x - \cos x = 0$

$\cos x = -\frac{1}{2}$

$\cos x = \cos\frac{2\pi}{3}$

$x = \pm\frac{2\pi}{3} \quad (2\pi)$

$\cos x = \cos\frac{2\pi}{3}$

$x = \pm\frac{2\pi}{3} \quad (2\pi)$

$2\sin x \cos x - \cos x = 0$

$\cos x\,(2\sin x - 1) = 0$

$\cos x = 0$ or $\sin x = \frac{1}{2}$

$x = \pm\frac{\pi}{2}\ (2\pi)$

$x = \frac{\pi}{6}\ (2\pi)$ or $\frac{5\pi}{6}\ (2\pi)$