

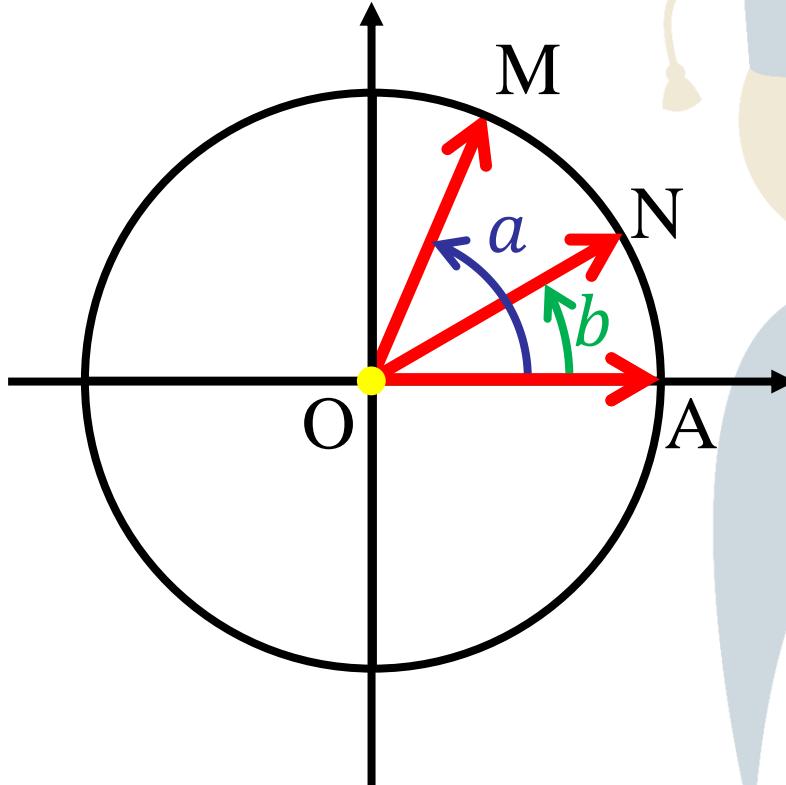




Trigonometric formulas

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Addition formulas



\widehat{AM} is an arc of measure a .

\widehat{AN} is an arc of measure b .

The coordinates of M are $(\cos a ; \sin a)$

The coordinates of N are $(\cos b ; \sin b)$

$$\cos(a - b) = ? ?$$

$$\begin{aligned} \cos(a - b) &= \frac{\overrightarrow{OM} \cdot \overrightarrow{ON}}{|OM| \times |ON|} = \frac{xx' + yy'}{1 \times 1} \\ &= \cos a \cos b + \sin a \sin b \end{aligned}$$

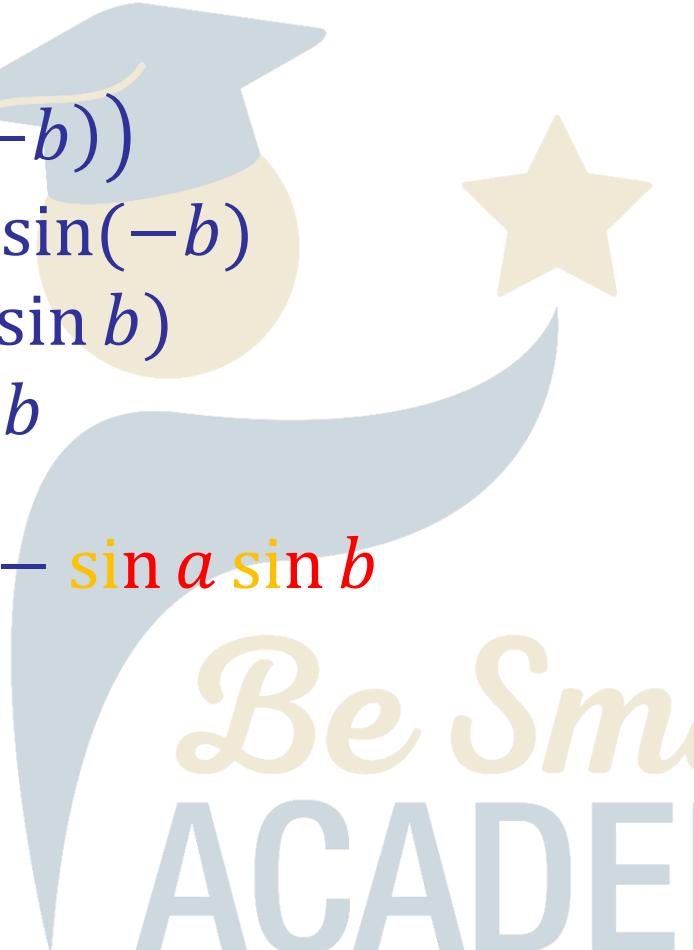
$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

Addition formulas

$\cos(a + b) = ??$

$$\begin{aligned}\cos(a + b) &= \cos(a - (-b)) \\&= \cos a \cos(-b) + \sin a \sin(-b) \\&= \cos a \cos b + \sin a (-\sin b) \\&= \cos a \cos b - \sin a \sin b\end{aligned}$$

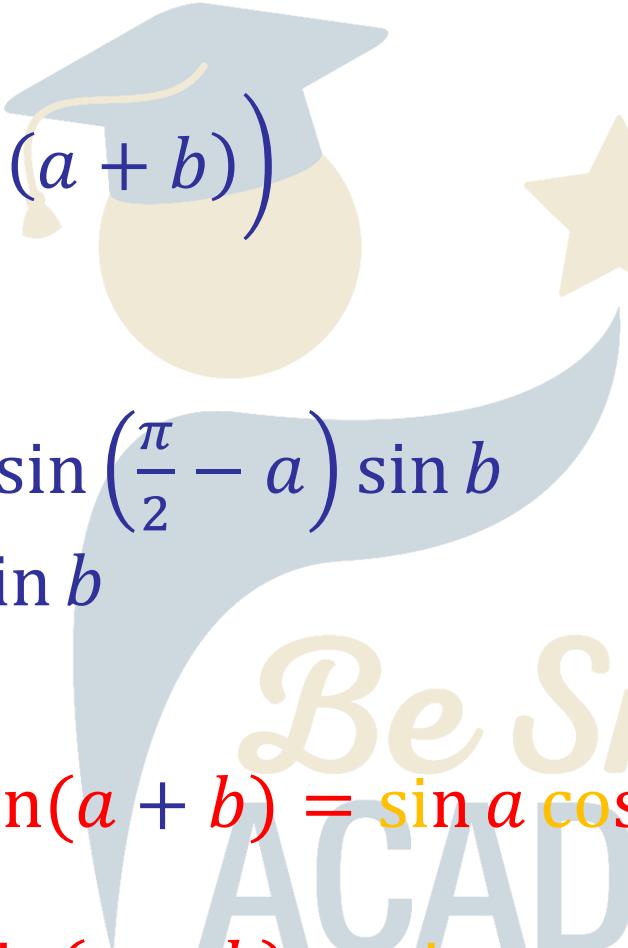
$\cos(a + b) = \cos a \cos b - \sin a \sin b$



Addition formulas

$$\sin(a + b) = ??$$

$$\begin{aligned}
 \sin(a + b) &= \cos\left(\frac{\pi}{2} - (a + b)\right) \\
 &= \cos\left(\left(\frac{\pi}{2} - a\right) - b\right) \\
 &= \cos\left(\frac{\pi}{2} - a\right)\cos b + \sin\left(\frac{\pi}{2} - a\right)\sin b \\
 &= \sin a \cos b + \cos a \sin b
 \end{aligned}$$



$$\sin(a - b) = ??$$

$$\begin{aligned}
 \sin(a - b) &= \sin(a + (-b)) \\
 &= \sin a \cos(-b) + \cos a \sin(-b) \\
 &= \sin a \cos b + \cos a (-\sin b) \\
 &= \sin a \cos b - \cos a \sin b
 \end{aligned}$$

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$$\begin{aligned}
 \sin(a + b) &= \sin a \color{blue}{\cos b} + \color{red}{\cos a} \sin b \\
 \sin(a - b) &= \sin a \color{blue}{\cos b} - \color{red}{\cos a} \sin b
 \end{aligned}$$

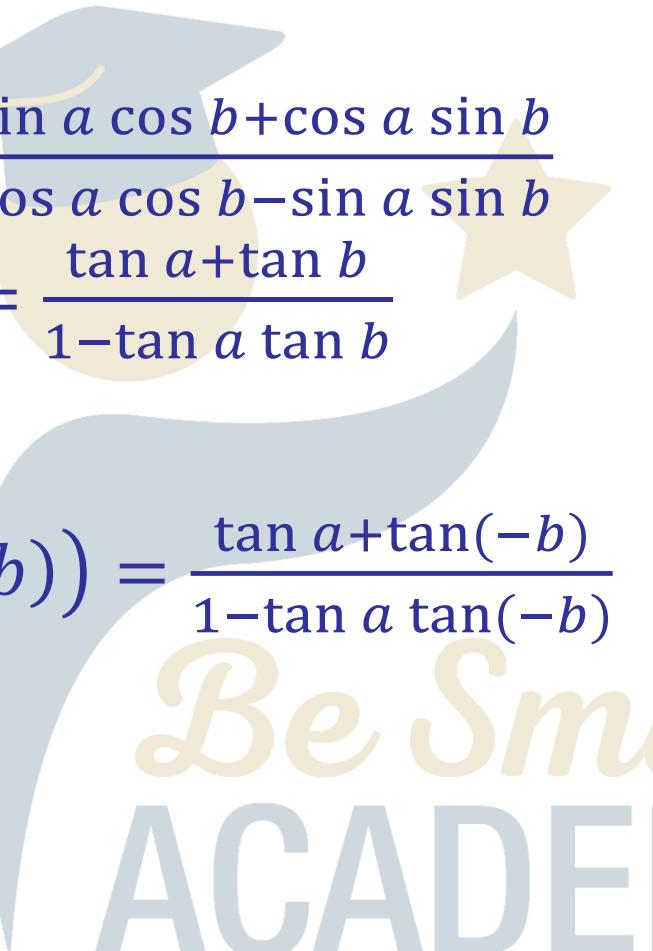
Addition formulas

$$\tan(a + b) = ??$$

$$\begin{aligned}\tan(a + b) &= \frac{\sin(a+b)}{\cos(a+b)} = \frac{\sin a \cos b + \cos a \sin b}{\cos a \cos b - \sin a \sin b} \\ &= \frac{\cos a \cos b(\tan a + \tan b)}{\cos a \cos b(1 - \tan a \tan b)} = \frac{\tan a + \tan b}{1 - \tan a \tan b}\end{aligned}$$

$$\tan(a - b) = ??$$

$$\begin{aligned}\tan(a - b) &= \tan(a + (-b)) = \frac{\tan a + \tan(-b)}{1 - \tan a \tan(-b)} \\ &= \frac{\tan a - \tan b}{1 + \tan a \tan b}\end{aligned}$$



$$\begin{aligned}\tan(a + b) &= \frac{\tan a + \tan b}{1 - \tan a \tan b} \\ \tan(a - b) &= \frac{\tan a - \tan b}{1 + \tan a \tan b}\end{aligned}$$

Example

$$\cos\left(\frac{\pi}{12}\right) = ??$$

$$\begin{aligned} \cos\left(\frac{\pi}{12}\right) &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \cos\frac{\pi}{3} \cos\frac{\pi}{4} + \sin\frac{\pi}{3} \sin\frac{\pi}{4} \\ &= \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4} \end{aligned}$$

$$\sin\left(\frac{\pi}{12}\right) = ??$$

$$\begin{aligned} \sin\left(\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \sin\frac{\pi}{3} \cos\frac{\pi}{4} - \cos\frac{\pi}{3} \sin\frac{\pi}{4} \\ &= \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4} \end{aligned}$$

$$\tan\left(\frac{\pi}{12}\right) = ??$$

$$\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$\begin{aligned} &= \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{3} \tan\frac{\pi}{4}} = \frac{\sqrt{3}-1}{1+\sqrt{3}\times 1} = \\ &\frac{\sqrt{3}-1}{1+\sqrt{3}} = \frac{4-2\sqrt{3}}{-2} = -2 + \sqrt{3} \end{aligned}$$

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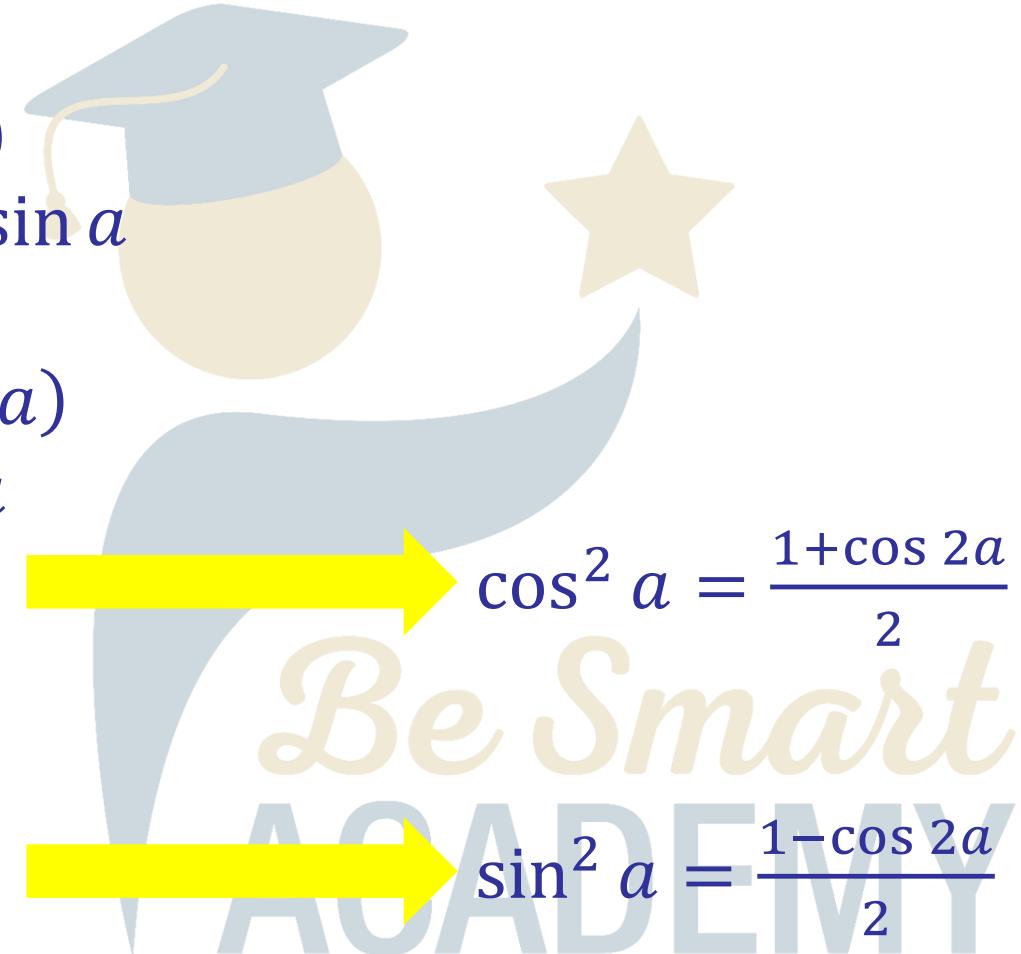
$$\cos\left(\frac{11\pi}{12}\right) = ??$$

$$\begin{aligned} \cos\left(\frac{11\pi}{12}\right) &= \cos\left(\pi - \frac{\pi}{12}\right) = \\ &- \cos\frac{\pi}{12} = -\frac{\sqrt{2}+\sqrt{6}}{4} \end{aligned}$$

Double angle formulas

$\cos(2a) = ??$

$$\begin{aligned}
 \cos(2a) &= \cos(a + a) \\
 &= \cos a \cos a - \sin a \sin a \\
 &= \cos^2 a - \sin^2 a \\
 &= \cos^2 a - (1 - \cos^2 a) \\
 &= \cos^2 a - 1 + \cos^2 a \\
 &= 2 \cos^2 a - 1 \\
 &= 2(1 - \sin^2 a) - 1 \\
 &= 2 - 2 \sin^2 a - 1 \\
 &= 1 - 2 \sin^2 a
 \end{aligned}$$



Double angle formulas

$\sin(2a) = ??$

$$\begin{aligned}\sin(2a) &= \sin(a + a) \\ &= \sin a \cos a + \cos a \sin a \\ &= 2 \sin a \cos a\end{aligned}$$

$\tan(2a) = ??$

$$\begin{aligned}\tan(2a) &= \tan(a + a) \\ &= \frac{\tan a + \tan a}{1 - \tan a \tan a} = \frac{2 \tan a}{1 - \tan^2 a}\end{aligned}$$



Example

Consider a real number a such that $\cos a = -\frac{3}{5}$

Calculate $\cos(2a)$; $\sin(2a)$ and $\tan(2a)$

$$\cos(2a) = 2 \cos^2 a - 1 = 2 \left(\frac{9}{25}\right) - 1 = -\frac{7}{25}$$

$$\sin^2(2a) = 1 - \cos^2(2a) = 1 - \frac{49}{625} = \frac{576}{625} \text{ so}$$

$$\sin(2a) = \pm \sqrt{\frac{576}{625}} = \pm \frac{24}{25}$$

$$\tan(2a) = \frac{\sin(2a)}{\cos(2a)} = \frac{\pm \frac{24}{25}}{-\frac{7}{25}} = \pm \frac{24}{7}$$

Factoring or Linearization

$$\cos p + \cos q = ??$$

Let $p = a + b$ and $q = a - b$

$$\text{So } a = \frac{p+q}{2} \quad \& \quad b = \frac{p-q}{2}$$

$$\begin{aligned}\cos p + \cos q &= \cos(a + b) + \cos(a - b) \\ &= \cos a \cos b - \sin a \sin b + \cos a \cos b + \sin a \sin b \\ &= 2 \cos a \cos b = 2 \cos\left(\frac{p+q}{2}\right) \cos\left(\frac{p-q}{2}\right)\end{aligned}$$

$$\cos p - \cos q = ??$$

$$\begin{aligned}\cos p - \cos q &= \cos(a + b) - \cos(a - b) \\ &= \cos a \cos b - \sin a \sin b - (\cos a \cos b + \sin a \sin b) \\ &= \cos a \cos b - \sin a \sin b - \cos a \cos b - \sin a \sin b \\ &= -2 \sin a \sin b = -2 \sin\left(\frac{p+q}{2}\right) \sin\left(\frac{p-q}{2}\right)\end{aligned}$$

Factoring or Linearization

$$\sin p + \sin q = ??$$

Let $p = a + b$ and $q = a - b$

$$\text{So } a = \frac{p+q}{2} \quad \& \quad b = \frac{p-q}{2}$$

$$\begin{aligned}\sin p + \sin q &= \sin(a + b) + \sin(a - b) \\ &= \sin a \cos b + \cos a \sin b + \sin a \cos b - \cos a \sin b \\ &= 2 \sin a \cos b = 2 \sin\left(\frac{p+q}{2}\right) \cos\left(\frac{p-q}{2}\right)\end{aligned}$$

$$\sin p - \sin q = ??$$

$$\begin{aligned}\sin p - \sin q &= \sin(a + b) - \sin(a - b) \\ &= \sin a \cos b + \cos a \sin b - (\sin a \cos b - \cos a \sin b) \\ &= \sin a \cos b + \cos a \sin b - \sin a \cos b + \cos a \sin b \\ &= 2 \cos a \sin b = 2 \cos\left(\frac{p+q}{2}\right) \sin\left(\frac{p-q}{2}\right)\end{aligned}$$

Example

Solve the equation: $\sin x + \sin 2x + \sin 3x = 1 + \cos x + \cos 2x$

$$\sin x + \sin 2x + \sin 3x = \sin(2x) + 2 \sin 2x \cos x = \sin 2x (1 + 2 \cos x)$$

$$1 + \cos x + \cos 2x = \cos x + 2 \cos^2 x = \cos x (1 + 2 \cos x)$$

$$\sin 2x (1 + 2 \cos x) = \cos x (1 + 2 \cos x)$$

$$\sin 2x (1 + 2 \cos x) - \cos x (1 + 2 \cos x) = 0$$

$$(1 + 2 \cos x)(\sin 2x - \cos x) = 0$$

$$1 + 2 \cos x = 0 \text{ or } \sin 2x - \cos x = 0$$

$$\cos x = -\frac{1}{2} \quad 2 \sin x \cos x - \cos x = 0$$

$$\cos x = \cos \frac{2\pi}{3} \quad \cos x = \cos \frac{2\pi}{3} \quad \cos x (2 \sin x - 1) = 0$$

$$x = \pm \frac{2\pi}{3} (2\pi) \quad x = \pm \frac{2\pi}{3} (2\pi) \quad \cos x = 0 \quad \text{or} \quad \sin x = \frac{1}{2}$$

$$x = \pm \frac{\pi}{2} (2\pi) \quad x = \frac{\pi}{6} (2\pi) \text{ or } \frac{5\pi}{6} (2\pi)$$

